Contents

1	Gen	eral Properties of Lattices	1		
	1.1	Lattices in Real Vector Spaces	1		
	1.2	Lattices in Euclidean Spaces	4		
	1.3	Duality	7		
	1.4	Automorphism Groups	12		
	1.5	Bilinear and Quadratic Forms	17		
	1.6	Quadratic Forms	18		
	1.7	The Dictionary Relating Lattices and Quadratic Forms	20		
	1.8	Packings	24		
	1.9	More on Integral Lattices	26		
	1.10	Tensor Product and Exterior Powers	30		
	1.11	Notes on Chapter 1	33		
2	Geo	metric Inequalities	37		
	2.1	The Hadamard Inequality	37		
	2.2	The Hermite Inequality	39		
	2.3	The Mordell Inequality	41		
	2.4	Mahler's Compactness Theorem	43		
	2.5	Lattice Constants	45		
	2.6	Extreme Lattices for an Open Star Body	47		
	2.7	The Lattice Constant for a Convex Star Body	51		
	2.8	Generalizations of the Hermite Invariant	55		
	2.9	The HKZ Reduction	59		
	2.10	Exercises for Chapter 2	62		
	2.11	Notes on Chapter 2	63		
3	Peri	Perfection and Eutaxy			
	3.1	Symmetric Endomorphisms	67		
	3.2	Linear Forms on Spaces of Endomorphisms	73		
	3.3	Linear Inequalities	77		
	3.4	A Characterization of Extreme Lattices	78		
	3.5	Perfect Configurations	81		
	3.6	Eutactic Configurations and Extreme Lattices	84		
	3.7	The Lamination Process	02		

XVIII Contents

	3.8	Dual-Extreme Lattices	94
	3.9	Exercises for Chapter 3	100
	3.10	Notes on Chapter 3	105
4	Roo	ot Lattices	109
	4.1	The \mathbb{Z}^n Lattice	110
	4.2	The \mathbb{A}_n Lattice	110
	4.3	The \mathbb{D}_n Lattice	112
	4.4	The \mathbb{D}_n^+ Packing and the \mathbb{E}_8 Lattice	114
	4.5	The Lattices \mathbb{E}_7 and \mathbb{E}_6	117
	4.6	Graphs and Inclusions Between Root Lattices	120
	4.7	Perfection and Eutaxy	124
	4.8	Some Other Constructions for Root Lattices	125
	4.9	Residual Quadratic Forms	129
	4.10	Root Systems	131
	4.11	Exercises for Chapter 4	138
	4.12	Notes on Chapter 4	145
5	Lati	tices Related to Root Lattices	147
	5.1	The Coxeter-Barnes Lattices \mathbb{A}_n^r	147
	5.2	The Coxeter Lattices \mathbb{A}_n^r	
	5.3	Barnes's Lattices P_n	
	5.4	Craig's Difference Lattices	163
	5.5	Lattices Related to the \mathbb{D}_n Lattice	
	5.6	Unimodular Lattices	
	5.7	Around the Leech Lattice	177
	5.8	Exercises for Chapter 5	182
	5.9	Notes on Chapter 5	187
6	Low	v-Dimensional Perfect Lattices	189
	6.1	A Combinatorial Characterization of the \mathbb{A}_n Lattices	
	6.2	Perfect Lattices up to Dimension 4	
	6.3	Dual-Extreme Lattices up to Dimension 4	
	6.4	Perfect Lattices in Dimension 5	200
	6.5	Perfect Lattices in Dimensions 6 and 7	208
	6.6	Some Indications About 8-Dimensional Perfect Lattices	
	6.7	Exercises for Chapter 6	219
	6.8	Notes on Chapter 6	223
7	The	Voronoi Algorithm	227
	7.1	Voronoi Domains	$\frac{-}{227}$
	7.2	Contiguity	234
	7.3	Finiteness Results	237
	7.4	The Voronoi Graphs	
	7.5	<u>*</u>	241

	Contents XIX
	7.6 The Voronoi Algorithm in Dimension 4
	7.7 The Facets of \mathbb{D}_n and the 5-Dimensional Perfect Lattices 24
	7.8 Determination of the Contiguous Form
	7.9 Perfect Forms in Dimensions 6 and 7
	7.10 Exercises for Chapter 7
	7.11 Notes on Chapter 7
8	Hermitian Lattices
	8.1 Complex and Quaternionic Structures
	8.2 Hurwitz Lattices: Enlargements of \mathbb{D}_n
	8.3 Hurwitz Lattices: Around Dimension 16
	8.4 Eisenstein Lattices: A Construction of Barnes
	8.5 Eisenstein Lattices: The Coxeter–Todd Lattice
	8.6 A General Construction of Hermitian Lattices
	8.7 Quadratic Hermitian Structures
	8.8 Beyond Dimension 24
	8.9 Exercises for Chapter 8
	8.10 Notes on Chapter 8
9	The Configurations of Minimal Vectors
	9.1 Minimal-Equivalent Lattices
	9.2 Classes of Dimension $n \leq 3$
	9.3 Classification in Dimension 4
	9.4 Weakly Eutactic Lattices in a Minimal Class
	9.5 The Classification of Eutactic Lattices
	9.6 Perfect Pairs of Lattices
	9.7 Complements
	9.8 Exercises for Chapter 9
	9.9 Notes on Chapter 9
10	Extremal Properties of Families of Lattices 36.
	10.1 Some Elementary Results on Lie Groups 36
	10.2 Perfection and Eutaxy
	10.3 Extremality
	10.4 Minimal Classes
	10.5 Dual-Extreme Lattices
	10.6 The Rankin Invariants
	10.7 Exercises for Chapter 10
	10.8 Notes on Chapter 10
11	Group Actions
	11.1 Rational and Integral Representations 38
	11.2 <i>G</i> -Lattices
	11.3 G-extreme Lattices
	11.4 Cyclotomic Lattices 30

XX	Contents	
1111	Contents	

	11.5 Isodual Lattices, Modular Lattices, and Normal Lattices. 11.6 Normal Lattices. 11.7 Extreme Symmetric and Symplectic Lattices. 11.8 Isodual Lattices: Examples and Classification Results. 11.9 Rationality and Finiteness Questions. 11.10 Exercises for Chapter 11 11.11 Notes on Chapter 11	403 407 414 417 421
12	Cross-Sections 12.1 Embedding a Lattice in a Larger One 12.2 X-Rays of Lattices 12.3 Lattices with a Fixed Cross-Section 12.4 A Characterization of Relatively Extreme Lattices 12.5 Patchwork Lattices 12.6 Exercises for Chapter 12 12.7 Notes on Chapter 12	427 430 433 437 438 439
13	Extensions of the Voronoi Algorithm 13.1 Contiguity Relative to a Space of Symmetric Matrices 13.2 The Voronoi Algorithm Relative to a Space of Symmetric Matrices 13.3 Perfect G-Lattices 13.4 Two-Dimensional Centralizers 13.5 Cyclotomic Lattices 13.6 Lattices with a Fixed Section and Patchwork Lattices 13.7 Examples 13.8 Exercises for Chapter 13 13.9 Notes on Chapter 13	444 448 451 454 458 460 463 464
14	Numerical Data 14.1 Low-Dimensional Perfect Lattices. 14.2 Root Lattices 14.3 Eutactic Lattices up to Dimension 4 14.4 The Hermite Constant 14.5 Invariants Related to Duality 14.6 The Kissing Number.	467 468 469 472 474
15	Appendix 1: Semi-Simple Algebras and Quaternions 15.1 Semi-Simple Algebras 15.2 Quaternion Algebras 15.3 Algebraic Lattices over Dedekind Domains 15.4 Arithmetic in Separable Algebras 15.5 Number Fields 15.6 Quaternions Again 15.7 Ideal Class Set	479 481 482 484 485

		Contents	XXI
16	Appendix 2: Strongly Perfect Lattices		. 489
	16.1 Spherical Designs		. 489
	16.2 Strong Perfection		. 491
	16.3 An Infinite Series		. 492
	16.4 Modular Lattices		. 493
	16.5 Group Theory		. 494
	16.6 Designs on Grassmannian Varieties		. 495
\mathbf{Re}	ferences		. 497
${f Lis}$	t of Symbols		. 511
Ind	lex		. 517