## Contents

1 General Properties of Lattices ..... 1
1.1 Lattices in Real Vector Spaces ..... 1
1.2 Lattices in Euclidean Spaces ..... 4
1.3 Duality ..... 7
1.4 Automorphism Groups ..... 12
1.5 Bilinear and Quadratic Forms ..... 17
1.6 Quadratic Forms ..... 18
1.7 The Dictionary Relating Lattices and Quadratic Forms ..... 20
1.8 Packings ..... 24
1.9 More on Integral Lattices ..... 26
1.10 Tensor Product and Exterior Powers ..... 30
1.11 Notes on Chapter 1 ..... 33
2 Geometric Inequalities ..... 37
2.1 The Hadamard Inequality ..... 37
2.2 The Hermite Inequality ..... 39
2.3 The Mordell Inequality ..... 41
2.4 Mahler's Compactness Theorem ..... 43
2.5 Lattice Constants ..... 45
2.6 Extreme Lattices for an Open Star Body ..... 47
2.7 The Lattice Constant for a Convex Star Body ..... 51
2.8 Generalizations of the Hermite Invariant ..... 55
2.9 The HKZ Reduction ..... 59
2.10 Exercises for Chapter 2 ..... 62
2.11 Notes on Chapter 2 ..... 63
3 Perfection and Eutaxy ..... 67
3.1 Symmetric Endomorphisms ..... 67
3.2 Linear Forms on Spaces of Endomorphisms ..... 73
3.3 Linear Inequalities ..... 77
3.4 A Characterization of Extreme Lattices ..... 78
3.5 Perfect Configurations ..... 81
3.6 Eutactic Configurations and Extreme Lattices ..... 84
3.7 The Lamination Process ..... 92
3.8 Dual-Extreme Lattices ..... 94
3.9 Exercises for Chapter 3 ..... 100
3.10 Notes on Chapter 3. ..... 105
4 Root Lattices ..... 109
4.1 The $\mathbb{Z}^{n}$ Lattice ..... 110
4.2 The $\mathbb{A}_{n}$ Lattice ..... 110
4.3 The $\mathbb{D}_{n}$ Lattice ..... 112
4.4 The $\mathbb{D}_{n}^{+}$Packing and the $\mathbb{E}_{8}$ Lattice ..... 114
4.5 The Lattices $\mathbb{E}_{7}$ and $\mathbb{E}_{6}$ ..... 117
4.6 Graphs and Inclusions Between Root Lattices ..... 120
4.7 Perfection and Eutaxy ..... 124
4.8 Some Other Constructions for Root Lattices ..... 125
4.9 Residual Quadratic Forms ..... 129
4.10 Root Systems ..... 131
4.11 Exercises for Chapter 4 ..... 138
4.12 Notes on Chapter 4 ..... 145
5 Lattices Related to Root Lattices ..... 147
5.1 The Coxeter-Barnes Lattices $\mathbb{A}_{n}^{r}$ ..... 147
5.2 The Coxeter Lattices $\mathbb{A}_{n}^{r}$ ..... 153
5.3 Barnes's Lattices $P_{n}$ ..... 157
5.4 Craig's Difference Lattices ..... 163
5.5 Lattices Related to the $\mathbb{D}_{n}$ Lattice ..... 171
5.6 Unimodular Lattices ..... 174
5.7 Around the Leech Lattice ..... 177
5.8 Exercises for Chapter 5 ..... 182
5.9 Notes on Chapter 5 ..... 187
6 Low-Dimensional Perfect Lattices ..... 189
6.1 A Combinatorial Characterization of the $\mathbb{A}_{n}$ Lattices ..... 190
6.2 Perfect Lattices up to Dimension 4 ..... 194
6.3 Dual-Extreme Lattices up to Dimension 4 ..... 196
6.4 Perfect Lattices in Dimension 5 ..... 200
6.5 Perfect Lattices in Dimensions 6 and 7 ..... 208
6.6 Some Indications About 8-Dimensional Perfect Lattices ..... 212
6.7 Exercises for Chapter 6 ..... 219
6.8 Notes on Chapter 6 ..... 223
7 The Voronoi Algorithm ..... 227
7.1 Voronoi Domains ..... 227
7.2 Contiguity ..... 234
7.3 Finiteness Results ..... 237
7.4 The Voronoi Graphs ..... 238
7.5 Lattices Contiguous to $\mathbb{A}_{n}$ ..... 241
7.6 The Voronoi Algorithm in Dimension 4 ..... 242
7.7 The Facets of $\mathbb{D}_{n}$ and the 5 -Dimensional Perfect Lattices ..... 244
7.8 Determination of the Contiguous Form ..... 253
7.9 Perfect Forms in Dimensions 6 and 7. ..... 254
7.10 Exercises for Chapter 7 ..... 258
7.11 Notes on Chapter 7 ..... 260
8 Hermitian Lattices ..... 263
8.1 Complex and Quaternionic Structures ..... 263
8.2 Hurwitz Lattices: Enlargements of $\mathbb{D}_{n}$ ..... 268
8.3 Hurwitz Lattices: Around Dimension 16 ..... 274
8.4 Eisenstein Lattices: A Construction of Barnes ..... 280
8.5 Eisenstein Lattices: The Coxeter-Todd Lattice ..... 284
8.6 A General Construction of Hermitian Lattices ..... 292
8.7 Quadratic Hermitian Structures ..... 298
8.8 Beyond Dimension 24 ..... 306
8.9 Exercises for Chapter 8 ..... 310
8.10 Notes on Chapter 8 ..... 316
9 The Configurations of Minimal Vectors ..... 321
9.1 Minimal-Equivalent Lattices ..... 321
9.2 Classes of Dimension $n \leq 3$. ..... 329
9.3 Classification in Dimension 4 ..... 333
9.4 Weakly Eutactic Lattices in a Minimal Class ..... 339
9.5 The Classification of Eutactic Lattices ..... 343
9.6 Perfect Pairs of Lattices ..... 348
9.7 Complements ..... 354
9.8 Exercises for Chapter 9 ..... 357
9.9 Notes on Chapter 9 ..... 361
10 Extremal Properties of Families of Lattices ..... 363
10.1 Some Elementary Results on Lie Groups ..... 364
10.2 Perfection and Eutaxy ..... 366
10.3 Extremality ..... 368
10.4 Minimal Classes ..... 373
10.5 Dual-Extreme Lattices ..... 374
10.6 The Rankin Invariants ..... 376
10.7 Exercises for Chapter 10 ..... 379
10.8 Notes on Chapter 10 ..... 380
11 Group Actions ..... 383
11.1 Rational and Integral Representations ..... 383
11.2 $G$-Lattices ..... 385
11.3 $G$-extreme Lattices ..... 388
11.4 Cyclotomic Lattices ..... 393
11.5 Isodual Lattices, Modular Lattices, and Normal Lattices ..... 400
11.6 Normal Lattices ..... 403
11.7 Extreme Symmetric and Symplectic Lattices ..... 407
11.8 Isodual Lattices: Examples and Classification Results ..... 414
11.9 Rationality and Finiteness Questions ..... 417
11.10 Exercises for Chapter 11 ..... 421
11.11 Notes on Chapter 11 ..... 425
12 Cross-Sections ..... 427
12.1 Embedding a Lattice in a Larger One ..... 427
12.2 X-Rays of Lattices ..... 430
12.3 Lattices with a Fixed Cross-Section ..... 433
12.4 A Characterization of Relatively Extreme Lattices ..... 437
12.5 Patchwork Lattices ..... 438
12.6 Exercises for Chapter 12 ..... 439
12.7 Notes on Chapter 12 ..... 440
13 Extensions of the Voronoi Algorithm ..... 443
13.1 Contiguity Relative to a Space of Symmetric Matrices ..... 444
13.2 The Voronoi Algorithm Relative to a Space of Symmetric Matrices ..... 448
13.3 Perfect $G$-Lattices ..... 451
13.4 Two-Dimensional Centralizers ..... 454
13.5 Cyclotomic Lattices ..... 458
13.6 Lattices with a Fixed Section and Patchwork Lattices ..... 460
13.7 Examples ..... 463
13.8 Exercises for Chapter 13 ..... 464
13.9 Notes on Chapter 13 ..... 465
14 Numerical Data ..... 467
14.1 Low-Dimensional Perfect Lattices ..... 467
14.2 Root Lattices ..... 468
14.3 Eutactic Lattices up to Dimension 4 ..... 469
14.4 The Hermite Constant ..... 472
14.5 Invariants Related to Duality ..... 474
14.6 The Kissing Number ..... 476
15 Appendix 1: Semi-Simple Algebras and Quaternions ..... 479
15.1 Semi-Simple Algebras ..... 479
15.2 Quaternion Algebras ..... 481
15.3 Algebraic Lattices over Dedekind Domains ..... 482
15.4 Arithmetic in Separable Algebras ..... 484
15.5 Number Fields ..... 485
15.6 Quaternions Again ..... 486
15.7 Ideal Class Set ..... 487
16 Appendix 2: Strongly Perfect Lattices ..... 489
16.1 Spherical Designs ..... 489
16.2 Strong Perfection ..... 491
16.3 An Infinite Series ..... 492
16.4 Modular Lattices ..... 493
16.5 Group Theory ..... 494
16.6 Designs on Grassmannian Varieties ..... 495
References ..... 497
List of Symbols ..... 511
Index ..... 517

