Contents

1 Projective Geometry 1
- 1.1 Projective Spaces 2
  - 1.1.1 Definitions and First Properties 2
  - 1.1.2 Plane Projective Geometry 11
- 1.2 Homogeneous Coordinates 15
  - 1.2.1 Definition. Simplices 15
  - 1.2.2 Coordinate Transformations. The Projective Linear Group 20
  - 1.2.3 Inhomogeneous Projective Coordinates 21
  - 1.2.4 The Projective Linear Group over a Field 22
- 1.3 Collineations 24
  - 1.3.1 Collinear Maps 25
  - 1.3.2 The Main Theorem of Projective Geometry 32
  - 1.3.3 The Group of Auto-Collineations 35
- 1.4 Cross Ratio and Projective Maps 40
  - 1.4.1 The Group Aut $\boldsymbol{P}^1$ 41
  - 1.4.2 The Cross Ratio 42
  - 1.4.3 Projective Maps 46
  - 1.4.4 Harmonic Position 51
  - 1.4.5 Staudt's Main Theorem 53
  - 1.4.6 Projective Equivalence of Collinear Maps. Involutions .. 55
- 1.5 Affine Geometry from the Projective Viewpoint 59
- 1.6 Duality 65
  - 1.6.1 Duality in Plane Incidence Geometry 66
  - 1.6.2 Projective and Algebraic Duality 67
  - 1.6.3 Projective Pencil Geometries 71
  - 1.6.4 Dual Maps 73
- 1.7 Correlations 75
  - 1.7.1 Definition. Canonical Correlation 76
  - 1.7.2 Correlative Maps 77
  - 1.7.3 F-Correspondences and σ-Biforms 80

1.8 Symmetric Auto-Correlative Maps 84
  1.8.1 Null Systems and Polar Maps 84
  1.8.2 Equivalence of Auto-Correlative Maps 86
  1.8.3 Classification of Null Systems 87
  1.8.4 Linear Line Complexes 89
1.9 Polarities and Quadrics 92
  1.9.1 σ-Hermitean Biforms 92
  1.9.2 Classification of Polar Maps 94
  1.9.3 The Real Polar Maps 96
  1.9.4 The Complex Polar Maps 96
  1.9.5 The Quaternionic Polar Maps 98
  1.9.6 Quadrics 101
  1.9.7 Polar Maps of Projective Lines 107
  1.9.8 Tangents and Tangent Subspaces 109
  1.9.9 Dualization: Coquadrics 111
1.10 Restrictions and Extensions of Scalars 112
  1.10.1 Hopf Fibrations 112
  1.10.2 Complex Structures 115
  1.10.3 Quaternionic Structures 117
  1.10.4 Projective Extensions 119
  1.10.5 The Projective K-Geometry of $\boldsymbol{P}_L^n$ 123
  1.10.6 K-Classification of Projective L-Subspaces 126
  1.10.7 Extensions of Null Systems, Polar Maps, and Quadrics . 128

2 Cayley-Klein Geometries 133
2.1 The Classical Groups 133
  2.1.1 The Linear and the Projective Groups 134
  2.1.2 The Projective Isotropy Group of a Correlation 136
  2.1.3 The Symplectic Groups 137
  2.1.4 The Orthogonal Groups 139
  2.1.5 The Unitary Groups 141
  2.1.6 The Quaternionic Skew Hermitean Polarities 143
  2.1.7 $\boldsymbol{SL}(n, K)$ is Generated by Transvections. 145
2.2 Vector Spaces with Scalar Product 148
  2.2.1 Vector, Projective, and Affine Geometries 149
  2.2.2 Subspaces 150
  2.2.3 E. Witt's Theorem 151
  2.2.4 Properties of Isotropic Subspaces 151
  2.2.5 The Proof of E. Witt's Theorem 154
  2.2.6 Transitivity Results 157
  2.2.7 Neutral Vector Spaces 159
  2.2.8 Tensors and Volume Functions 161
  2.2.9 The General Vector Product 163
  2.2.10 Adjoint Linear Maps 165
  2.2.11 Properties of the Root Subspaces 172

2.3 The Projective Geometry of a Polarity 176
  - 2.3.1 The Quadric of a Polarity 176
  - 2.3.2 Efficiency 179
  - 2.3.3 Orthogonal Geometry. Reflections 182

2.4 Invariants of Finite Configurations 186
  - 2.4.1 $\boldsymbol{PG}_n$-Congruence of Finite Point Sequences 186
  - 2.4.2 Orbits of Points. Normalized Representatives 189
  - 2.4.3 Invariants of Point Pairs 191
  - 2.4.4 Real Orthogonal Geometries 196
  - 2.4.5 Projective Orthogonal Geometries for Arbitrary Fields 198
  - 2.4.6 Plane Cone Sections 203

2.5 Spherical and Elliptic Geometry 207
  - 2.5.1 Spherical as a Covering of Elliptic Geometry 207
  - 2.5.2 Distance and Angle 211
  - 2.5.3 The Law of Cosines and the Triangle Inequality 217
  - 2.5.4 Excess, Curvature, and Surface Area 221
  - 2.5.5 Spherical Trigonometry 227
  - 2.5.6 The Metric Geometry of Elliptic Space 232
  - 2.5.7 Angle between Subspaces and Distance of Great Spheres 234
  - 2.5.8 Quadrics 238

2.6 Hyperbolic Geometry 243
  - 2.6.1 Models of Hyperbolic Space 245
  - 2.6.2 Distance and Angle 256
  - 2.6.3 Distance and Angles as Cross Ratios 260
  - 2.6.4 The Hyperbolic Law of Cosines and Hyperbolic Metric . 266
  - 2.6.5 Hyperbolic Trigonometry 272
  - 2.6.6 Hyperspheres, Equidistants, and Horospheres 277
  - 2.6.7 Stationary Angles 283
  - 2.6.8 Quadrics 291
  - 2.6.9 Pictures of Hyperbolic Quadrics 309

2.7 Möbius Geometry 314
  - 2.7.1 Spheres in Möbius Space 314
  - 2.7.2 Pairs of Subspheres 319
  - 2.7.3 Cross Ratios and the Riemann Sphere 331
  - 2.7.4 Möbius Invariants and Euclidean Invariants 341
  - 2.7.5 Three-Dimensional Möbius Geometry 344
  - 2.7.6 Orbits, Cyclids of Dupin, and Loxodromes 349

2.8 Projective Symplectic Geometry 358
  - 2.8.1 Symplectic Transvections 359
  - 2.8.2 Subspaces 361
  - 2.8.3 Triangles 363
  - 2.8.4 Skew Lines 366
  - 2.8.5 Symmetric Bilinear Forms and Quadrics 374

2.9 Transformation Groups: Results and Problems 385

Basic Notions from Algebra and Topology 403
A.1 Notations 403
A.2 Linear Algebra 404
A.3 Transformation Groups 404
A.4 Topology 407

References 417

Index 423