

Table of Contents

1	Linear Systems and the Time Optimal Control Problem.....	1
1.1	Linear Systems	1
1.2	Accessibility Set and Controllability	1
1.3	Controllability and Feedback Classification in the Autonomous Case	2
1.3.1	Controllability	2
1.3.2	Linear Classification	3
1.3.3	Feedback Classification.....	5
1.3.4	Stabilization	6
1.4	Controllability in the Nonautonomous Case	7
1.4.1	Application	8
1.5	Time Optimal Control for Linear Systems	9
1.5.1	Notations	9
1.6	Feedback Equivalence to a Linear System.....	15
1.6.1	Problem Statement	15
1.7	Time Minimal Synthesis	17
1.7.1	Notion of Synthesis	17
1.7.2	The General Algorithm in the Plane	21
	Exercises	24
2	Optimal Control for Nonlinear Systems	27
2.1	A Short Visit into the Classical Calculus of Variations	27
2.1.1	Statement of the Problem in the Holonomic Case	27
2.1.2	Hamiltonian Equations.....	29
2.1.3	Hamilton-Jacobi-Bellman Equation	30
2.1.4	Euler-Lagrange Equations and Characteristics of the HJB Equation	31
2.1.5	Second Order Conditions	31
2.1.6	The Accessory Problem and the Jacobi Equation	32
2.1.7	Conjugate Point and Local Morse Theory	33
2.1.8	Scalar Riccati Equation	34
2.1.9	Local C^0 Minimizer - Extremal Field - Hilbert Invariant Integral.....	35
2.2	Optimal Control and the Calculus of Variations	36

X Table of Contents

2.2.1	Problem Statement	36
2.2.2	The Augmented System	37
2.2.3	Related Problems	38
2.2.4	Optimal Control and the Classical Calculus of Variations	38
2.2.5	Singular Trajectories and the Weak Maximum Principle	39
2.2.6	First and Second Variations of $E^{x_0, T}$	39
2.2.7	Geometric Interpretation of the Adjoint Vector	42
2.2.8	The Weak Maximum Principle	42
2.2.9	Abnormality	43
2.2.10	The Weak Maximum Principle and Euler-Lagrange Equation	43
2.2.11	Comparison with the Calculus of Variations	44
2.2.12	LQ-Control and the Weak Maximum Principle	44
2.3	Pontryagin's Maximum Principle (PMP)	45
2.4	Filippov Existence Theorem	52
2.4.1	Comments about the Existence Theorem	52
2.5	Dynamic Programming and the Maximum Principle	53
	Exercises	56
3	Geometric Optimal Control.....	65
3.1	Introduction to Symplectic Geometry	65
3.2	Geometric Classification of Extremals in one Dimensional Problems of Calculus of Variations.....	69
3.2.1	Problem Statement and Preliminaries	69
3.2.2	Singularity Analysis	71
3.2.3	Generic Classification near Σ_1	72
3.2.4	Classification and Existence of Optimal Solutions.....	75
3.3	Time Minimum Control Problem	76
3.4	Determination of the Singular Extremals	77
3.4.1	Hamiltonian Formalism and Singular Extremals	78
3.5	Geometric Classification of Extremals near Σ_i	79
3.5.1	Normal Switching Points	79
3.5.2	The Fold Case	80
3.6	Comparaison Between the Calculus of Variations and the Time Minimum Problem for Affine Systems. The Role of Sin- gular Extremals	82
3.7	The Fuller Phenomenon	82
3.7.1	Fuller Example.....	82
3.8	Geometry of the Time Optimal Control in the Plane	84
3.8.1	Preliminaries	84
3.8.2	Generic points	85
3.8.3	Singular arc	86
3.8.4	Elliptic Case - The Concept of Conjugate Point	88
	Exercises	94

4 Singular Trajectories and Feedback Classification	97
4.1 Classification of Affine Systems	99
4.1.1 Computations of Singular Controls	99
4.1.2 Singular Trajectories and Feedback Classification	102
4.1.3 Time Optimality and Feedback Classification	105
4.2 Singular Trajectories and the Problem of Classification of Distributions	108
4.2.1 Preliminaries	108
4.2.2 Local Classification in Dimension 3, with $rank D = 2$	109
4.3 Feedback Classification and Analytic Geometry	111
4.3.1 Preliminaries	111
4.3.2 Critical Hamiltonians and Symbols	112
Exercises	113
5 Controllability - Higher Order Maximum Principle - Legendre-Clebsch and Goh Necessary Optimality Conditions	117
5.1 Some Notations and Formulas from Differential Geometry	117
5.1.1 Controllability with Piecewise Constant Controls	119
5.2 Integrating Distributions	120
5.2.1 Preliminaries	120
5.2.2 Frobenius Theorem	120
5.2.3 Nagano-Sussmann Theorem	121
5.2.4 C^∞ -Counter Example	122
5.3 Nonlinear Controllability and Chow Theorem	122
5.4 Poisson Stability and Controllability	124
5.4.1 Application	125
5.5 Controllability and Enlarge Technique	125
5.5.1 Problem Statement	125
5.6 Evaluation of the Accessibility Set	127
5.6.1 Legendre-Clebsch Condition	129
5.7 The Multi-Inputs Case: Goh Condition	131
5.8 The Concept of Rigidity - Strong Legendre Clebsch and Goh Conditions as Necessary Conditions for Rigidity	132
5.8.1 Intrinsic Second Variation - Morse Index	133
Exercises	134
6 The Concept of Conjugate Points in the Time Minimal Control Problem for Singular Trajectories, C^0-Optimality	143
6.1 Single-Input Case	143
6.1.1 Preliminaries	143
6.1.2 Feedback Semi-Normal Forms in the Hyperbolic and Elliptic Cases	146
6.1.3 LQ-Model	147
6.1.4 Approximation of the End-Point Mapping Using the LQ-Model in the Elliptic Case	148

XII Table of Contents

6.1.5	Conclusion	153
6.1.6	The Exceptional Case.....	153
6.1.7	Conclusion	155
6.1.8	Comparison of the Elliptic-Hyperbolic Case and the Exceptional Case	156
6.2	Applications	157
6.2.1	Time Optimal Synthesis for Planar System	157
6.2.2	Example in Dimension 3 and Connection with the Hamilton-Jacobi-Bellman Equation	157
6.3	The Case in \mathbb{R}^3 . Intrinsic Computations of Conjugate Points - Connection with the Time Minimal Synthesis Problem. The Concept of Curvature	160
6.3.1	Euler-Lagrange Equation	160
6.3.2	Geometric Interpretation	161
6.3.3	Intrinsic Computation	162
6.3.4	The Concept of Curvature	162
6.3.5	Conjugate Points and Time Minimal Synthesis	163
6.4	Connection with the Liu-Sussmann Example	164
6.4.1	Preliminaries	164
6.4.2	Conclusion	166
6.4.3	Statement and Proof of Liu-Sussmann Result	166
6.4.4	Conclusion	167
6.4.5	Connection with the Sub-Riemannian Geometry	168
7	Time Minimal Control of Chemical Batch Reactors and Sin- gular Trajectories.....	171
7.1	Introduction	171
7.2	Mathematical Model of Chemical Batch Reactors and De- scription of the Control Problem	172
7.2.1	Chemical Kinetics	172
7.2.2	Control Device	174
7.2.3	The Optimal Control Problem	175
7.2.4	Projected Problem	175
7.3	Singular Extremals - Curvature - Conjugate Points	176
7.3.1	Projected System.....	177
7.4	Time Minimal Synthesis for Planar Systems in the Neighbor- hood of a Terminal Manifold of Codimension One	180
7.4.1	Problem Statement	180
7.4.2	Assumption C1	181
7.4.3	The Generic C1 Case	181
7.4.4	The Generic C1 Flat Case	181
7.4.5	The Case C1 of Codimension One	182
7.4.6	Generic Hyperbolic Cases	185
7.4.7	Generic Exceptional Case	188
7.4.8	Generic Flat Exceptional Case	189

Table of Contents XIII

7.5	Global Time Minimal Synthesis	191
7.5.1	Preliminaries	191
7.5.2	Singular Arc	191
7.5.3	Regular Arcs	192
7.5.4	Optimal Synthesis in the Neighborhood of N	193
7.5.5	Switching Rules	193
7.5.6	Optimal Synthesis	194
7.6	State Constraints due to the Temperature	195
7.6.1	Preliminaries	195
7.6.2	Optimal Synthesis	198
7.7	The Problem in Dimension 3	199
7.7.1	Preliminaries	199
7.7.2	Stratification of N by the Optimal Feedback Synthesis	199
7.7.3	Orientation Principle	200
7.7.4	Switching Rules	201
7.7.5	Local Classification near the Target	203
7.7.6	Focal Points	206
8	Generic Properties of Singular Trajectories	209
8.1	Introduction and Notations	209
8.2	Determination of the Singular Extremals with Minimal Order	210
8.3	Statement of the First Generic Property	211
8.4	Geometric Interpretation and the General Concept of Order .	211
8.5	Proof of Theorem 25	213
8.5.1	Partially Algebraic and Semi-Algebraic Fiber Bundles .	217
8.5.2	Coordinate Systems on $P(d, N)$	218
8.5.3	Evaluation of Codimension of the $\mathcal{F}(N)$	218
8.5.4	End of the Proof of Theorem 25	222
8.6	Genericity of Codimension One Singularity	222
8.7	Proof of Theorem 26	222
8.7.1	The “Bad” Set for Theorem 26	223
8.7.2	Evaluation of the Codimension of $B_c(N, q)$	224
8.8	Singularities of the Singular Flow of Minimal Order	226
8.8.1	Preliminaries (see Sect. 4.1.3)	226
8.8.2	Local Classification near C	227
8.8.3	Local Classification near $S \setminus C$	228
8.8.4	The Quadratic Case	229
	Exercises	231
9	Singular Trajectories in Sub-Riemannian Geometry	233
9.1	Introduction	233
9.2	Generalities About SR-Geometry	234
9.2.1	Definition	234
9.2.2	Optimal Control Theory Formulation	234

XIV Table of Contents

9.2.3	Computations of the Extremals and Exponential mapping	235
9.3	Research Program in SR-Geometry	239
9.3.1	Classification	239
9.3.2	Singularity Theory of the Exponential Mapping and of the Distance Function	239
9.4	Privileged Coordinates and Graded Normal Forms	240
9.4.1	Regular and Singular Points	240
9.4.2	Adapted and Privileged Coordinates	241
9.4.3	Nilpotent Approximation	242
9.4.4	Graded Approximation	242
9.5	The Contact Case of Order -1 or the Heisenberg-Brockett Example	242
9.5.1	The Contact Case in \mathbb{R}^3	242
9.5.2	Symmetry Group in the Heisenberg Case	243
9.5.3	Heisenberg SR-Geometry and the Dido Problem	243
9.5.4	Geodesics	244
9.5.5	Conjugate Points	245
9.5.6	Sphere and Wave Front	245
9.5.7	Conclusion About the SR-Heisenberg Geometry	246
9.6	The Generic Contact Case	247
9.6.1	Normal Forms in the Contact Case	247
9.6.2	Generic Conjugate Locus	248
9.7	The Martinet Case	249
9.7.1	Preliminaries	249
9.7.2	Normal Form	250
9.7.3	Orthonormal Frame	253
9.7.4	Graded Normal Form	253
9.7.5	Geodesics	254
9.7.6	Riemannian Metric on the Plane (x, y)	255
9.7.7	Asymptotic Foliation Associated to the Normal Form at Order 0	256
9.7.8	Properties of the Asymptotic Foliations	257
9.7.9	Integrable Case of Order 0 and Elliptic Integrals	258
9.7.10	The Martinet Flat Case	262
9.8	Estimates of the Sphere and of the Wave Front near the Ab- normal Direction in the Flat Case and $\exp - \ln$ Category	266
9.8.1	Intersection of $S(0, r)$ with the Cut Locus	266
9.9	Conclusion Deduced from the Martinet SR-Flat Geometry Concerning the role of Abnormal Geodesics in SR-Geometry	268
9.9.1	Behaviors of the Normal Geodesics near the Abnormal Direction - Geodesic C^0 -Rigidity	268
9.9.2	Nonproperness of the First Return Mapping R_1 and Geometric Consequence	269

Table of Contents XV

9.10 Cut Locus in Martinet SR-Geometry	271
Exercises	273
10 Micro-Local Resolution of the Singularity near a Singular Trajectory - Lagrangian Manifolds and Symplectic Stratifications	281
10.1 Introduction	281
10.2 Lagrangian Manifolds	281
10.3 Application to Classical Calculus of Variations	282
10.4 Singularity Theory of the Generating Function - Generating Family	283
10.5 Application to Optimal Control Theory and to SR-Geometry	285
10.5.1 The SR-Normal Case	285
10.5.2 Jacobi Fields	287
10.5.3 Reduction Algorithm in the SR-Case	288
10.6 The Singular Case	289
10.6.1 A Geometric Remark	289
10.6.2 Heisenberg Case	289
10.6.3 Martinet Flat Case	289
10.7 Resolution of the Singularity in the Hyperbolic Case	292
10.7.1 Normal Form	292
10.7.2 Intrinsic Second-Order Derivative	293
10.7.3 Goh Transformation	294
10.8 Lagrangian Manifolds in the Hyperbolic Case	294
10.9 Examples	296
10.10 Resolution of the Singularity in the Exceptional Case	302
10.10.1 Normal Form - Intrinsic Derivative	302
10.10.2 The Operators	303
10.10.3 Isotropic Manifolds in the Exceptional Case	303
10.11 Micro-Local Analysis of SR-Martinet Ball of Small Radius - Lagrangian Stratification of the Martinet Sector	306
10.11.1 Preliminaries	306
10.11.2 The Smooth Abnormal Sector	307
10.11.3 The Smoothness of the Sphere in the Abnormal Direction	307
10.11.4 The Martinet Sector	309
10.11.5 Symplectic Stratifications	312
10.11.6 Transcendence of the Sector	312
10.11.7 The Micro-Local Analysis of the Whole Martinet Sphere	313
Exercises	314
11 Numerical Computations	321
11.1 Introduction	321
11.2 Numerical Algorithm	321
11.3 Applications	324

XVI Table of Contents

11.3.1 Contact Case in Dimension 3	324
11.3.2 The Martinet Flat Case	324
11.4 The Tangential Case	328
11.4.1 The Elliptic Case	329
11.4.2 The Hyperbolic Case	330
12 Conclusion and Perspectives	335
12.1 The Category of the Distance Function in SR-Geometry	335
12.2 SR-Distances and Control of the Oscillations of Solutions of Ordinary Differential Equations	335
12.3 Optimal Control Problems with Bounded State Variables	338
12.3.1 Maximum Principle with State Constraints	338
12.4 Singular Trajectories and Regularity of Stabilizing Feedback ..	339
12.5 Computations of Singular Trajectories	340
12.6 Computations of Conjugate Points	340
Exercises	341
References	343
Index	349